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Reference motion in deformable bodies under rigid body motion and vibration. Part I: theory

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Abstract

This paper examines the motions of reference systems linked to deformable bodies under simultaneously vibration and large translations and rotations. These motions depend on the particular type of linkage between the moving reference system and the deformable body, which is defined by the so-called *reference conditions*. When using the Rayleigh–Ritz method, the reference conditions also dictate the boundary conditions to be fulfilled by the shape functions used to describe the body's elasticity. This paper analyses three different types of reference conditions, namely: free linkage, rigid linkage and two-point linkage. It is shown that, moving reference frames only evolve at a constant velocity in the absence of external forces when the free linkage is used. The reference velocities for systems with a free linkage are designated *rigid body equivalent velocities* for the deformable body here. Such velocities can also be calculated under other types of reference conditions and are usually functions of the elastic and reference co-ordinates, and also of their derivatives. Rigid body equivalent velocities are useful for purposes such as estimating the trajectory of deformable bodies moving freely in space without the need to examine the deformations they undergo. Also, their calculation is required with a view to determining the kinematic restitution coefficient for deformable body collisions, which is dealt within Part II of this series.

1. Introduction

The dynamics of deformable bodies under large rotations and translations can be analyzed in terms of absolute co-ordinates [1–3] or by using a mixed set of reference and relative elastic co-ordinates [4–6]. When using absolute co-ordinates, they are referred to a single inertial reference system. This method is only appropriate when the bodies concerned undergo large deformations;

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with small deformations, the *floating frame of reference approach*, which uses the latter above-mentioned co-ordinate system, is more effective.

In the *floating frame of reference approach*, each deformable body is assigned a local co-ordinate system. A series of Cartesian co-ordinates and orientation co-ordinates or parameters is used to indicate the origins of the local co-ordinate systems and their orientations, respectively, with respect to a global co-ordinate system (inertial axes). The body's deformation is described by a set of elastic co-ordinates referred to the associated local system. Deformations in the body can be described by using various spatial discretization approaches including the Rayleigh–Ritz method and the finite element method. In both cases, the body's deformation at any time is assumed to be a linear combination of a series of shape functions the coefficients of which are the elastic co-ordinates. The shape functions used for this purpose should fulfill specific boundary conditions that arise from the type of linkage between the local co-ordinate system and the deformable body. The reference conditions specify the type of linkage and become boundary conditions that must be obeyed by the shape functions at the linkage point(s).

The best way of selecting the reference conditions and hence their associated shape functions for describing the deformation has been examined by many authors. Agrawal [7] recommends using the mean axis conditions as reference in order to minimize dynamic coupling between the elastic and reference co-ordinates. The mean axis conditions minimize the kinetic energy associated to the elastic co-ordinates. Also, in the absence of forces and moments acting on the system, such conditions imply that the total momentum and angular momentum due to the deformation are always zero. The modes of vibration of free bodies in space are appropriate shape functions for the mean axis conditions. Kim and Haug [8] proposed the use of local reference systems rigidly attached to a point in the deformable body and of modes of vibration combined with static modes of deformation to describe the deformation. Static modes of vibration, which they called *constraint modes*, are obtained by applying unit forces or torques to specific nodes selected using finite element models. Meirovitch and Kwak [9] showed that using the method of floating reference systems with a single family of admissible shape functions results in poor convergence characteristics because a single family of functions does not allow one to satisfy the dynamic boundary conditions of the deformable body. They used *quasi*-comparison functions (viz. combinations of families of shape functions obtained under different boundary conditions in such a way that a linear combination would meet any dynamic boundary condition) to circumvent this shortcoming. Schwertassek et al. [10,11] classified local reference systems into *tangent frames*, *chord frames* and *mean* or *Buckens frames*. Measured deformations decrease in the sequence tangent frames > chord frames > mean frames. Because the method of floating reference systems is restricted to small deformations, mean frames extend their scope to the greatest possible extent. However, previous authors have shown that the modes of deformation of the free body in space used with this type of frame are inappropriate for estimating stress as they usually fail to fulfill the dynamic boundary conditions. For this reason, they proposed using static modes in addition to dynamic modes as shape functions to describe deformations.

This paper analyses the dynamics of planar motion in elastic beams by use of the *floating frame of reference approach*. Specifically, it examines the relationship between the reference co-ordinates and the resulting motions in the equivalent rigid bodies. To this end, local axes with a free linkage to the deformable body (mean axes), axes rigidly linked to a point in the deformable body (tangent axes) and axes supported on two points of the body (chord axes) were used. The following section provides a kinematic description and the dynamic equations derived using the method of floating reference systems. Section 3 analyses the different inertial terms present in the equations depending on the particular reference conditions and the point of linkage to the origin of the floating reference system. Section 4 defines the rigid body equivalent velocities for deformable bodies. Section 5 describes the application of the previous results and demonstrates their usefulness with the problem of the throw and free flight of a javelin. Finally, Section 6 summarizes the most interesting aspects of the study and its most salient conclusions.

2. Kinematic description

In the proposed formulation, the position of a point P in a deformable body i with respect to the global co-ordinate system depicted in Fig. 1 is given by

$$\mathbf{R}^p = \mathbf{R} + \mathbf{A}\left(\bar{\mathbf{r}}^P + \bar{\mathbf{r}}_f^P\right),$$

$$\bar{\mathbf{r}}_f^P = \mathbf{\Psi}\left(\bar{\mathbf{r}}^P\right)\mathbf{q}_f, \tag{1}$$

where $\mathbf{R}$ is the position vector for the origin of the local co-ordinate system with respect to the global system, $\mathbf{A}$ the matrix of rotation from the local co-ordinate system to the global one, which depends on angle θ, $\bar{\mathbf{r}}^p$ the position of P with respect to the local co-ordinate system when the body is undeformed, $\bar{\mathbf{r}}_f^P$ the elastic displacement vector; $\boldsymbol{\psi}$ a matrix containing the shape functions for elastic displacements; and $\mathbf{q}_f$ the elastic co-ordinate vector.

In the planar motion, the equations of motion for the deformable body are

$$\mathbf{M(q)\ddot{q}} + \mathbf{Kq} = \mathbf{Q}_v + \mathbf{Q}_{ext},$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{R} \\ \theta \\ \mathbf{q}_f \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{R\theta} & \mathbf{m}_{Rf} \\ & \mathbf{m}_{\theta\theta} & \mathbf{m}_{\theta f} \\ Sym & & \mathbf{m}_{ff} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} \\ Sym & & \mathbf{K}_{ff} \end{bmatrix}, \quad \mathbf{Q}_v = \begin{bmatrix} (\mathbf{Q}_v)_R \\ (\mathbf{Q}_v)_\theta \\ (\mathbf{Q}_v)_f \end{bmatrix}, \tag{2}$$

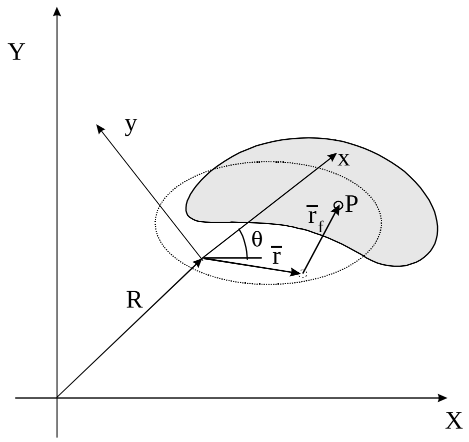

Fig. 1. Position of a point P in a deformable body.

where vector $\mathbf{q}$ contains co-ordinates $\mathbf{R}$, θ and $\mathbf{q}_f$, $\mathbf{M}$ is the mass matrix (co-ordinate-dependent), $\mathbf{K}$ is stiffness matrix, $\mathbf{Q}_v$ the quadratic velocity vector and $\mathbf{Q}_{ext}$ the vector of generalized external forces acting on the body. Damping forces are assumed to be zero.

3. Analysis of inertial terms

The mass matrix is given by [4]

$$\mathbf{M} = \begin{bmatrix} m\mathbf{I} & \mathbf{A}_\theta\left[m\bar{\mathbf{r}}_G + \int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f\right] & \mathbf{A}\int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V \\ & I_0 + \mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f + 2\int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f & \int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}\,\mathrm{d}V + \mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}\,\mathrm{d}V \\ Sym & & \int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V \end{bmatrix}, \quad (3)$$

where subscripts G and O denote the centre of gravity of the body and the origin of the local co-ordinate system, respectively; $\mathbf{A}_\theta$ is the partial derivative of the rotation matrix, $\mathbf{A}$, with respect to angle θ, $\mathbf{I}$ the identity matrix and $\tilde{\mathbf{I}}$ is defined as

$$\tilde{\mathbf{I}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (4)$$

The quadratic velocity vector is given by [4]

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{A}\left[m\bar{\mathbf{r}}_G + \int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f\right]\dot{\theta}^2 - 2\dot{\theta}\mathbf{A}_\theta\left[\int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\dot{\mathbf{q}}_f\right] \\ -2\dot{\theta}\dot{\mathbf{q}}_f^{\mathrm{T}}\left[\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f + \int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\bar{\mathbf{r}}_P\,\mathrm{d}V\right] \\ \dot{\theta}^2\left[\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f + \int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\right] + 2\dot{\theta}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}\,\mathrm{d}V\dot{\mathbf{q}}_f \end{bmatrix}. \quad (5)$$

When the modes of vibration of the free flexible body in space $\boldsymbol{\psi}^f$ (where superscript f denotes free boundary conditions) are used as shape functions, some terms in the previous equations cancel. This type of shape function can be used provided a free linkage between the local co-ordinate system and the flexible body is adopted as reference conditions. In this type of linkage, the origin of the local co-ordinate system is linked to no material point of the body; also, elastic stress on the boundary of the body is always zero as this is a boundary condition imposed on the shape functions. As a rule, the momentum, $\mathbf{p}_f$, and the net angular momentum with respect to the origin of the local co-ordinate system L_{fO} due to elastic motions are obtained from the following expression:

$$\mathbf{p}_f = \int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\dot{\mathbf{q}}_f, \quad L_{fO} = \left[\int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}\,\mathrm{d}V + \mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f\right]\dot{\mathbf{q}}_f. \quad (6)$$

When using modes of vibration of a free body in space, these quantities should be zero as the body is assumed to be under no external forces or moments. Consequently,

$$\int_V \rho\boldsymbol{\Psi}^f\,\mathrm{d}V = 0, \quad \int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}^f\,\mathrm{d}V + \mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{f\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\Psi}^f\,\mathrm{d}V\mathbf{q}_f = 0. \quad (7)$$

If the elastic body is a beam, then

$$\int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \tilde{\mathbf{I}} \mathbf{\Psi}^f \, \mathrm{d}V = 0 \tag{8}$$

because longitudinal and transverse elastic displacements are assumed to be uncoupled. Therefore, from Eq. (7) it follows that

$$\int_V \rho \bar{\mathbf{r}}_P^{\mathrm{T}} \tilde{\mathbf{I}} \mathbf{\Psi}^f \, \mathrm{d}V = 0. \tag{9}$$

Consequently, with this type of shape function, the mass matrix cross-terms between the elastic and reference co-ordinates cancel as $\mathbf{m}_{Rf} = \mathbf{m}_{\theta f} = \mathbf{0}$, so translations in the local co-ordinate system are completely uncoupled with elastic motions. However, the orientation co-ordinate θ is not uncoupled with elastic motions as the inertial quadratic velocity terms, $(\mathbf{Q}_v)_\theta$ and $(\mathbf{Q}_v)_f$, are non-zero and depend on $\dot{\theta}$, $\mathbf{q}_f$ and $\dot{\mathbf{q}}_f$.

If, in addition to using shape functions such as $\boldsymbol{\psi}^f$, points O and G are made to coincide in the undeformed position (i.e., if the centre of gravity of the elastic body is used as the origin of the local co-ordinate system), then translations and rotations in the system are also uncoupled. Under these conditions, the mass matrix and the quadratic velocity vector are given by

$$\mathbf{M} = \begin{bmatrix} m\mathbf{I} & \mathbf{0} & \mathbf{0} \\ & I_G + \mathbf{q}_f^{\mathrm{T}} \int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \mathbf{q}_f + 2 \int_V \rho \bar{\mathbf{r}}_P^{\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \mathbf{q}_f & \mathbf{0} \\ Sym & & \int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \end{bmatrix}, \tag{10}$$

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{0} \\ -2\dot{\theta} \dot{\mathbf{q}}_f^{\mathrm{T}} \left[\int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \mathbf{q}_f + \int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \bar{\mathbf{r}}_P \mathrm{d}V \right] \\ \dot{\theta}^2 \left[\int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \mathbf{q}_f + \int_V \rho \bar{\mathbf{r}}_P^{\mathrm{T}} \mathbf{\Psi}^f \, \mathrm{d}V \right] + 2\dot{\theta} \int_V \rho \mathbf{\Psi}^{f\mathrm{T}} \tilde{\mathbf{I}} \mathbf{\Psi}^f \, \mathrm{d}V \dot{\mathbf{q}}_f \end{bmatrix}, \tag{11}$$

where I_G is the moment of inertia with respect to point G in the undeformed configuration.

4. Rigid body equivalent velocity

In solving some problems of practical interest it is important to know the rigid body equivalent motions of flexible bodies. As shown above, rigid body translations can be directly calculated by using the above-described local systems and shape functions. However, alternative types of local systems and shape functions also allow such rigid body equivalent motions to be determined, albeit in a more elaborate manner. Rigid body equivalent velocities can usually be calculated by assuming the elastic body concerned to be a system consisting of infinite particles. In this way, the rigid body equivalent velocities, $\mathbf{V}_{RB}$, can be taken to be the total momentum, $\mathbf{p}$, divided into the body mass:

$$\mathbf{p} = \mathbf{m}_{RR} \dot{\mathbf{R}} + \mathbf{m}_{R\theta} \dot{\theta} + \mathbf{m}_{Rf} \dot{\mathbf{q}}_f,$$

$$\mathbf{V}_{RB} = \frac{\mathbf{p}}{m}. \tag{12}$$

If a $\boldsymbol{\psi}^f$ shape function is used, then

$$\mathbf{V}_{RB} = \dot{\mathbf{R}}. \tag{13}$$

Similarly, the equivalent angular velocity of a deformable body, ω_{RB}, can be obtained by calculating the angular momentum with respect to its centre of gravity and dividing it by its moment of inertia. The angular momentum for the deformable body with respect to the origin of the locales axes, O, can be obtained from

$$L_O = \mathbf{m}_{\theta R}\dot{\mathbf{R}} + \mathbf{m}_{\theta\theta}\dot{\theta} + \mathbf{m}_{\theta f}\dot{\mathbf{q}}_f. \tag{14}$$

Also, the angular momentum with respect to the centre of gravity, G, of the deformable body can be obtained by adding L_0 to the vector product of the momentum, $\mathbf{p}$, by the vector joining O and G [12], which can be expressed in matrix form as

$$L_G = L_O + \mathbf{p}^{\mathrm{T}}\tilde{\mathbf{I}}\mathbf{A}\left(\bar{\mathbf{r}}^G + \bar{\mathbf{r}}_f^G\right),$$

$$\bar{\mathbf{r}}_f^G = \frac{1}{m}\int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f. \tag{15}$$

The resulting expression for L_G is rather complex but simplifies to the following when O and G coincide in the undeformed position:

$$L_G = \left[\mathbf{m}_{\theta\theta} - \frac{1}{m}\mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\mathrm{d}V\int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f\right]\dot{\theta} + \left[\mathbf{m}_{\theta f} - \frac{1}{m}\mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\mathrm{d}V\tilde{\mathbf{I}}\int_V \rho\boldsymbol{\Psi}\,\mathrm{d}V\right]\dot{\mathbf{q}}_f. \tag{16}$$

As can be seen, L_G is independent of $\mathbf{R}$. This is the case even when O and G do not coincide. The rigid body equivalent angular velocity of the deformable body can thus be obtained from

$$\omega_{RB} = \frac{L_G}{I_G}. \tag{17}$$

If a shape function of the $\boldsymbol{\psi}^f$ type is used, and O and G are made to coincide in the undeformed position, then Eq. (17) reduces to

$$\omega_{RB} = \frac{\mathbf{m}_{\theta\theta}\dot{\theta}}{I_G} = \frac{I_G + \mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f + 2\int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f}{I_G}\dot{\theta}. \tag{18}$$

Therefore, unlike the reference velocities $\dot{\mathbf{R}}$ given by Eq. (13), the derivative of θ with respect to time is not constant with the proposed reference conditions and shape functions —not even in the absence of external torques. This is the result of $\mathbf{m}_{\theta\theta}$ not being constant and of the fact that, as shown below, the product $\mathbf{m}_{\theta\theta}\theta$ should remain constant. The terms

$$\mathbf{q}_f^{\mathrm{T}}\int_V \rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f + 2\int_V \rho\bar{\mathbf{r}}_P^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}V\mathbf{q}_f, \tag{19}$$

that appear in Eq. (18), represent the change in moment of inertia undergone by the deformable body by effect of the deformation. With small deformations, such terms are negligible relative to I_G; under these conditions—which are those used in the method of floating reference systems

$$\omega_{RB} \approx \dot{\theta}. \tag{20}$$

In summary, using free linkages between the local co-ordinate system and the deformable body, making O and G coincide in the undeformed position and using the modes of vibration of the free body as shape functions results in the translations of the local co-ordinate system coinciding with the rigid body equivalent motions of the deformable body. In addition, the rotations in the reference systems roughly coincide with the rigid body equivalent rotations of the deformable body. With other reference conditions or shape functions, these equivalent motions are determined by using Eqs. (12) and (18).

5. Analysis of the throw and flight of a javelin

The implications of using different reference conditions and the usefulness of the rigid body equivalent velocities are analyzed in this section by examining the throw and free flight of a deformable javelin. The following three types of analysis are performed for this purpose:

1. With mean frame reference conditions (free linkage) and the modes of vibration for a free beam as shape functions.
2. With tangent frame reference conditions (fixed linkage) and the modes of vibration for a cantilever beam as shape functions.
3. With chord frame reference conditions (two-points linkage) and the modes of vibration for a simply supported beam as shape functions.

The different floating axes used are shown in Fig. 2. Initially, all axes coincided with the central section of the javelin. Also, the javelin was assumed to undergo transverse deformation but negligible axial deformation. The modes of vibration were obtained as continuous functions —no finite elements were used but the results would have been comparable. The javelin was assumed to be an elastic beam of uniform cross-section; also, the air resistance was not modelled. In summary, the javelin was assumed to be a beam of length $l = 2\,\mathrm{m}$, mass $m = 1\,\mathrm{kg}$, sectional inertia $I = 3 \times 10^{-8}\,\mathrm{m}^4$ and Young's modulus 5 GPa. A value of 9.81 $\mathrm{m/s^2}$ was adopted for the acceleration due to gravity.

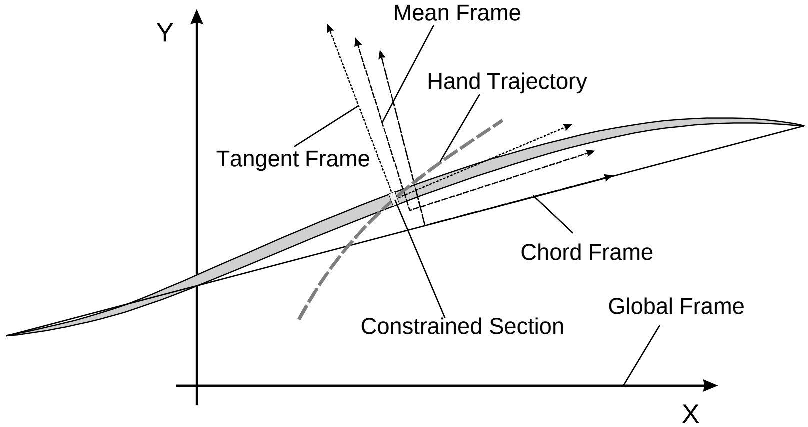

Fig. 2. Reference conditions.

In order to simulate the javelin, the trajectory and orientation of the central section were subjected to a series of kinematic constraints for over a preset period. In this way, the action of the thrower's hand was represented by assuming that the javelin was grasped at its centre of gravity. The constraints on the motion of the central section and its orientation were applied over a period $t_f = 1\,\mathrm{s}$. The constraint equations used were

$$
\begin{aligned}
R_X^S(t) &= R_X - \sin(\theta)\mathbf{\Psi}(\xi = 0)\mathbf{q}_f = \tfrac{1}{2}at^2, \\
R_Y^S(t) &= R_Y + \cos(\theta)\mathbf{\Psi}(\xi = 0)\mathbf{q}_f = -c\left(\frac{a}{2}\right)^2 t^4 + \tfrac{b}{2}at^2, \quad 0<t<t_f, \\
\theta^S(t) &= \theta + \frac{1}{l}\frac{\mathrm{d}\mathbf{\Psi}(\xi = 0)}{\mathrm{d}\xi}\mathbf{q}_f = a\tan\left(-cat^2 + b\right)
\end{aligned} \tag{21}
$$

where superscript S denotes the central section of the javelin; $\xi \in [-0.5, 0.5]$ is the non-dimensional variable along the beam; and a, m and k are three parameters that define the parabolic path followed by the hand. Such parameters were assigned the values $a = 15\,\mathrm{m/s^2}$, $b = 1.4$ and $c = 0.035\,\mathrm{m^{-1}}$.

If the javelin were a rigid body, from the kinematic solution to the skew throw, the evolution of the velocity components of the centre of gravity, the angular velocity and the trajectory of the centre of gravity would be given by

$$
\begin{aligned}
&V_X(t) = V_{X0}, V_Y(t) = V_{Y0} - gt, \omega(t) = \omega_0, \\
&Y = \frac{1}{V_{X0}}\left(V_{Y0} - \frac{g}{2V_{X0}}X\right)X,
\end{aligned} \tag{22}
$$

where subscript 0 denotes velocities at the end of the throw (i.e., while the constraints imposed through Eq. (21) applied). As suggested by Eq. (13), the evolution of the derivatives at the origin of the mean axes (solution 1) coincides with those resulting from Eq. (22). Such is also the case with the trajectory of the origin. As predicted by Eq. (20), the evolution of the derivative of the orientation of the mean axes coincides roughly with that resulting from Eq. (22). With tangent or chord frames (solution 2 and 3, respectively), however, the derivatives of the position of the origin of the reference frames, their trajectories and the derivatives of their orientation co-ordinates do not coincide with those provided by Eq. (22). With tangent or chord frames, the rigid body equivalent velocities can be obtained from Eqs. (12) and (18). These quantities, which are functions of the elastic and reference co-ordinates, coincide with the values yielded by Eq. (22).

This example illustrates one case where determining the rigid body equivalent velocities is of special interest. Once known, they can be used in Eq. (22) to determine the maximum height reached or distance travelled by the javelin during its free flight following release by the thrower. These data can be obtained without the need to analyze the javelin deformation during the free flight.

The throw and free flight of the javelin were simulated using the three above-described types of analysis and six elastic co-ordinates for all solutions. The modes of vibration used with each of the three solutions examined were as follows [13]:

$$
\begin{aligned}
\psi_i^f(\xi) = {} & \sin\left(\mu_i^f(\xi+0.5)\right) + \sinh\left(\mu_i^f(\xi+0.5)\right) \\
& - \frac{\cos\mu_i^f + \cosh\mu_i^f}{\sin\mu_i^f + \sinh\mu_i^f}\left(\cos\left(\mu_i^f(\xi+0.5)\right) + \cosh\left(\mu_i^f(\xi+0.5)\right)\right), \\
\psi_i^c(\xi) = {} & \sin\left(\mu_i^c(\xi+0.5)\right) - \sinh\left(\mu_i^c(\xi+0.5)\right) \\
& + \frac{\cos\mu_i^c + \cosh\mu_i^c}{\sin\mu_i^c - \sinh\mu_i^c}\left(\cos\left(\mu_i^c(\xi+0.5)\right) - \cosh\left(\mu_i^c(\xi+0.5)\right)\right), \\
\psi_i^s(\xi) = {} & \sin(i\pi(\xi+0.5)),
\end{aligned} \tag{23}
$$

where superscripts f, c and s denote a free beam, a cantilever beam and a simply supported beam, respectively. With the former two sets of modes, constants μ_i were obtained from the following non-linear algebraic equations:

$$
\begin{aligned}
\cos\mu_i^f \cosh\mu_i^f &= 1, \\
\cos\mu_i^c \cosh\mu_i^c &= -1.
\end{aligned} \tag{24}
$$

The equations of motion were numerically integrated using the stabilization method of Baumgarte [5,6] to ensure fulfillment of the constraint equations; a Runge–Kutta single-step explicit formula was also employed [14]. The equations were solved using Matlab software.

Fig. 3 shows a portion of the trajectory of the origin of the different local frames in the zone near the maximum height of the path. Fig. 4 shows the X component of the velocity of the origin of the different local frames. Only with free frames did such a velocity remain constant; with the other two types, there were deviations from the mean velocity. Fig. 5 shows the Y component of the velocity of the origin of the different local frames. Similarly, only with free frames was a uniformly (negatively) accelerated motion represented. Fig. 6 shows the angular velocities for the different local frames. Although not strictly constant with free frames (as shown by using Eq. (18)), this quantity only changed in the fourth significant place during the simulation.

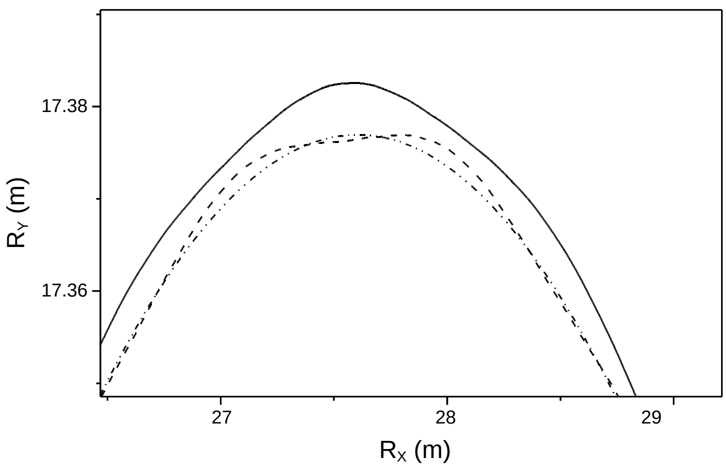

Fig. 3. Trajectory of the local reference frames: – – – – , mean axis; - - - - , chord axis; —— fixed axis.

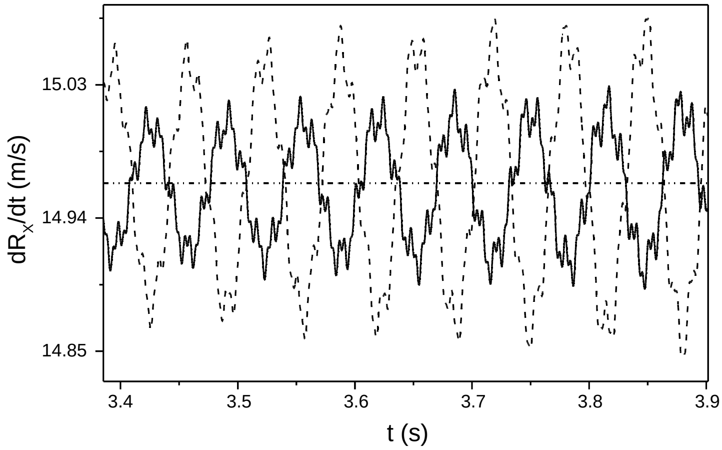

Fig. 4. X component of the velocity of the different local reference frames: – - – -, mean axis; - - - -, chord axis; ——, fixed axis.

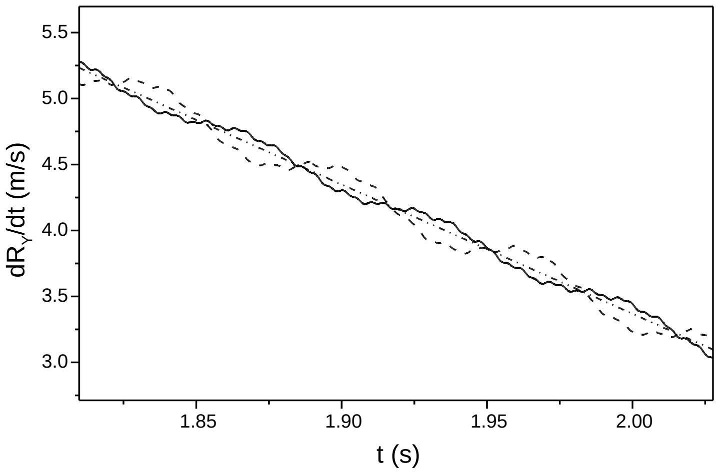

Fig. 5. Y component of the velocity of the different local reference frames: – - – -, mean axis; - - - -, chord axis, ——, fixed axis.

Although the translation velocities provided by Eqs. (2) differed little from the rigid body values, the angular velocities in solutions 2 and 3 exhibited substantial differences (see Figs. 4–6).

It should be noted that the velocities for solutions 2 and 3 obtained from Eqs. (12)–(18) coincided with the reference velocities of the solution found with free frames and hence with the rigid body equivalent velocities.

6. Summary and conclusions

In this work, the motions associated with reference systems attached to the deformable bodies used by *the floating frame of reference approach* were examined. Such motions are frequently

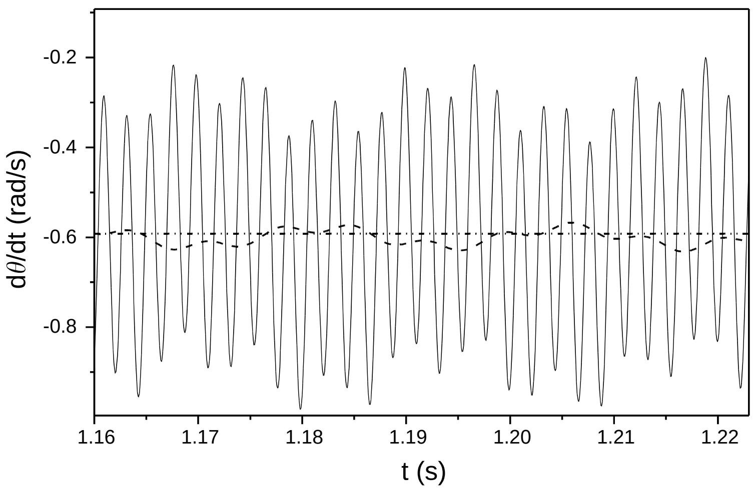

Fig. 6. Angular velocity of the different local reference frames: –-–-, mean axis; - - - -, chord axis; ——, fixed axis.

assumed to represent rigid body motions; however, this assumption can lead to spurious results. The relationship between the reference motions and the rigid body motions depend on the reference conditions used for the local axes and also on the type of shape functions employed to describe deformation in the bodies.

The use of floating axes supported on the centre of gravity of the deformable body, a free linkage to such a body and the modes of vibration for the free body in space substantially simplifies the dynamic equations involved. In addition, the reference velocities coincide with the rigid body equivalent velocities. This is strictly true for translation velocities but only approximate for angular velocities. Expressions for calculating the rigid body velocities with all types of reference conditions and shape functions are provided here. Such expressions rely on the assumption that the deformable body consists of a set of infinite particles.

Rigid body equivalent velocities are extremely useful with a view to exploiting the knowledge of the motion that bodies would have if they behaved as rigid bodies. In many cases, such motions can be determined in a direct manner. This situation is illustrated with the problem of the throw and flight of a javelin. As shown in Part II of this paper, the rigid body equivalent velocities are required to calculate the kinematic coefficient of restitution for the impact of deformable bodies.

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